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BULLETIN OF TUIT:

MANAGEMENT AND COMMUNICATION TECHNOLOGIES



ISSN:2181-1083 DOI: Date:2025-11-03
Begulov U.U. № 3(28)2025

UDC 517.956.4, 519.633.2

NUMERICAL MODELING OF A PARABOLIC EQUATION IN A MOVING MEDIUM WITH DOUBLE NONLINEARITY AND NONHOMOGENEITY

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Abstract— the qualitative properties of the heat diffusion equation

$$\rho(t)\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(u^{\sigma}\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right)-\rho(t)\frac{\partial u}{\partial x}$$

in a medium with double nonlinearity and exponentially varying density are studied. Theorems on global and asymptotic solutions are proved, and the resulting consequences are presented. Using the obtained results as an initial approximation, numerical results were obtained by applying the shooting method in the Python programming language. To analyze these results, graphs were generated for the nonlinear heat diffusion process.

Keywords: *numerical solution, self-similar solution, shooting method, existence of global solution, asymptotics.*

1. Introduction

In this article, we consider the Cauchy problem for a doubly nonlinear parabolic-type equation with variable density in the domain $Q=\{(t,x)\mid t>0, x\in R\}$. The problem is given by the following:

$$\rho(t)\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(u^{\sigma}\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right)-\rho(t)\frac{\partial u}{\partial x},\ (t,x)\in Q \tag{1}$$

$$u(0,x)=u_0(x)\geq 0, x\in R \tag{2}$$

Here, $\sigma>0, p\geq 2$ are given parameters, and $\rho(t)=e^{\alpha t}$ is the density function.

It is well known that equation (1) is degenerate, and it may not possess a classical solution when $u=0$ or $\frac{\partial u}{\partial x}=0$. Therefore, the solution of problem (1)–(2) is considered as a generalized solution in the class $u^{\sigma}\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\in C\left(R\times(0,+\infty)\right),\ u>0$, and this solution satisfies equation (1) in the sense of an integral identity.

2. Research Methodology

In the works of many researchers, the problem (1) – (2) has been studied in

particular cases, and it has been shown that such models describe various physical and biological processes [3–8].

In particular, in [8], the following quasilinear heat equation was investigated under certain boundary conditions:

$$\frac{\partial \varphi(u)}{\partial t} = \frac{\partial}{\partial x}\left(K(u)\frac{\partial u}{\partial x} \right).$$

A numerical solution to this equation was presented using finite difference schemes. It was explained that the number of iterations, i.e., the convergence of the process, depends on the choice of the interpolation formula for thermal conductivity. Methods for applying these formulas and the corresponding numerical results were also presented.

In [9], the authors considered a one-dimensional nonlinear parabolic equation that models diffusion, filtration, and heat conduction processes, under three different types of boundary conditions:

$$\frac{\partial T}{\partial t} = div\left(D(u)\frac{\partial T}{\partial x} \right).$$

Within this work, theorems analogous to the Cauchy-Kovalevskaya theorem were proven. Several examples were discussed, numerical results were obtained, and corresponding graphs were presented and analyzed based on these results.

In works [10–13], the following equation:

$$\rho(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^{\sigma}\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x} \right) + \varepsilon\cdot\gamma(t)\rho(x)u^{\beta}$$

was studied as a heat diffusion equation in a doubly nonlinear, nonhomogeneous medium (with $\rho(x) = \{e^{x}, e^{\alpha x}\}$), under conditions where $\varepsilon = \{-1, 0, 1\}$ and $\gamma(t) = \{t^{\lambda}, (T+t)^{m}\}$. In these cases, a self-similar (automodel) solution of the equation was found, and a theorem was proved concerning whether the solution serves as an upper or lower solution. Additionally, the asymptotic behavior of the solution was analyzed.

In the work by Aripov M.M. and Sadullaeva Sh.A. [14], the following equation:

$$Au \equiv -\rho(x)\frac{\partial u}{\partial t} + L(n, m, p)u + \varepsilon\gamma(t)\rho(x)u^{\beta} = 0,$$

$$u(0, x) = u_0(x), x \in R^N.$$

was studied as a doubly nonlinear reaction-diffusion type equation. The properties of both self-similar and approximately self-similar solutions were investigated. The influence of different parameter values on the dynamics of the process was identified. The existence of quenching solutions (solutions vanishing in finite time) was

proven, and their asymptotic behavior at infinity was described. Furthermore, the conditions for the global existence of the Cauchy problem solution were established. Here, $L(n,m,p)u = \nabla\left(|x^n|u^{m-1}|\nabla u|^{p-2}\nabla u\right)$, and $\beta \geq 1, n, p, m$ are given numerical parameters, $\nabla(\cdot) - grad_x(\cdot), 0 < \gamma(t) \in C(0,\infty), \varepsilon = \pm 1,$ and $\rho(x) = |x|^{\delta-2}$.

In [15], the asymptotic analysis of blow-up solutions for the following equation was presented:

$$\rho(x)u_t = (u^m)_{xx} \text{ in } Q = R \times R_+,$$

Here, $\rho(x) = |x|^{-\alpha}, e^{-x^2}, e^{-x}$.

In works [16–17], the authors studied nonlinear parabolic equations with source terms and nonhomogeneous density of the following form:

$$\rho(x)\frac{\partial u}{\partial t} = div\left(u^{m-1}|Du|^{p-2}Du\right) + u^{\beta} \text{ and}$$

$$\rho(x)\frac{\partial u}{\partial t} = div\left(u^{m-1}|Du|^{p-2}Du\right) + \rho(x)u^{\beta},$$

where

$$\rho(x) = |x|^{-n} \text{ or } \rho(x) = (1+|x|)^{-n}, n \geq 0.$$

They determined the conditions under which the solution to the Cauchy problem blows up in finite time.

3. Preliminary Concepts and Self-Similar Analysis

We seek the solution $u(t,x)$ in the following form:

$$u(t,x) = w\left(\tau(t), \gamma\left(\tau(t), x\right)\right) \tag{3}$$

We substitute the solution in this form into equation (1). As a result, we obtain the following equation:

$$\frac{\partial u}{\partial \tau} = \frac{\partial}{\partial \gamma}\left(w^{\sigma}\left|\frac{\partial w}{\partial \gamma}\right|^{p-2}\frac{\partial w}{\partial \gamma}\right) \tag{4}$$

Here,

$$\tau(t) = \int \frac{1}{\rho(t)}dt,$$

$$\gamma\left(\tau(t), x\right) = x - \int_0^{\tau(t)} \rho(s)ds.$$

To transition from equation (4) to a self-similar equation, we apply the substitution

$$w(\tau, \gamma) = f(\xi), \text{ where } \xi = \frac{\gamma}{\tau^{\frac{1}{p}}},$$

and obtain the following self-similar equation:

$$\frac{d}{d\xi}\left(f^{\sigma}\left|\frac{df}{d\xi}\right|^{p-2}\frac{df}{d\xi}\right) + \frac{\xi}{p}\frac{df}{d\xi} = 0 \tag{5}$$

Here, we consider nontrivial, non-negative solutions of equation (5) that satisfy the following conditions:

$$f(0)=M, M\in R_+,\ f(d)=0, 0<d<\infty \tag{6}$$

4. Estimate of solution

To solve problem (5) – (6), we use the comparison theorem from [3] and the method of model equations from [2], which allows us to obtain an approximate self-similar solution for problem (1) – (2). For this purpose, we seek the solution in the following form:

$$\overline{f}(\xi)=C\left(a-\xi^n\right)_+^m \tag{7}$$

where $a\geq 0, (d)_+=\max(0,d)$.

Using equation (7), we perform the necessary calculations for equation (5) and determine the parameters n, m and C:

$$n=\frac{p}{p-1},\ m=\frac{p-1}{\sigma+p-2}, C=\left[\frac{1}{n\cdot m}\left(\frac{1}{p}\right)^{\frac{1}{p-1}}\right]^m.$$

Theorem 1.

If the parameters of the solution sought in the form $\overline{f}(\xi)$ are as determined above for problem (1)–(2), and the condition

$$u(0,x)\leq u_+(0,x),\ \ x\in R$$

holds, then problem (1) – (2) admits a global solution defined in the domain Q ,and the following estimate is valid:

$$u(t,x)\leq u_+(t,x).$$

Here,

$$u_+(t,x)=\overline{f}(\xi),$$

$$n=\frac{p}{p-1},\ m=\frac{p-1}{\sigma+p-2}, C=\left[\frac{1}{n\cdot m}\left(\frac{1}{p}\right)^{\frac{1}{p-1}}\right]^m$$

is defined in this form.

Proof.

To prove this theorem, we introduce the following notation for equation (1):

$$Lu=-\rho(t)\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(u^\sigma\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right)-\rho(t)\frac{\partial u}{\partial x}$$

Given the parameters stated in the theorem and the condition $u(0,x)\leq u_+(0,x)$, the inequality $Lu_+(t,x)\leq 0$ holds. This shows that $u_+(t,x)$ is an upper solution of problem (1)–(2) according to the solution comparison principle. The solution has the following form:

$$u_+(t,x)=\left[\frac{1}{n\cdot m}\left(\frac{1}{p}\right)^{\frac{1}{p-1}}\right]^m\left(a-\left[\frac{x-\int_0^{\tau(t)}\rho(s)ds}{\left[\int\frac{1}{\rho(t)}dt\right]^{\frac{1}{p}}}\right]^n\right)^m$$

5. Asymptotics of the Compactly Supported Self-Similar Solution

Theorem 2.

For the compactly supported solutions of problem (5) – (6), the following asymptotic behavior holds as $\xi\to a^{\frac{p-1}{p}}$:

$$f\left(\xi\right)=\overline{f}\left(\xi\right)\left(1+o\left(1\right)\right)$$

(8)

Proof.

We seek the solution of equation (5) in the following form:

$$f\left(\xi\right)=\overline{f}\left(\xi\right)y\left(\eta\right)$$

(9)

Here, $\eta=-\ln\left(a-\xi^{\frac{p}{p-1}}\right)$ and

$\overline{f}\left(\xi\right)=C\left(a-\left(\frac{\gamma}{\tau^{\frac{1}{p}}}\right)^{n}\right)_{+}^{m}$ are defined in this form. To find the asymptotic behavior of the solution of equation (5), we analyze it near the point $\xi\to a^{\frac{p-1}{p}}$. Using the solution obtained in the form (9), and performing a number of calculations for equation (5), we derive the following equation for $y(\eta)$:

$$\frac{d}{d\eta}(Ly)+b_0(\eta)Ly+b_1(\eta)(\frac{dy}{d\eta}-\gamma_2 y)=0$$

(10)

Here,

$$b_0(\eta)=\frac{1}{n}\frac{e^{-\eta}}{a-e^{-\eta}}-m,\ b_1(\eta)=\frac{n^{1-p}}{C^{\sigma+p-2}p},$$

$$Ly=y^{\sigma}\left|\frac{dy}{d\eta}-my\right|^{p-2}\left(\frac{dy}{d\eta}-my\right).$$

We rewrite equation (10) in the following form:

$$\frac{d}{d\eta}(Ly)=-b_0(\eta)Ly-b_1(\eta)(\frac{dy}{d\eta}-\gamma_2 y)$$

(11)

We assume that $\xi\in[\xi_0,\xi_1), 0<\xi_0<\xi_1$. Moreover, the function $\eta(\xi)$ is a function defined on the interval $\xi\in[\xi_0,\xi_1)$ that possesses the following properties:

a) $\eta'(\xi)>0$,

b) $\eta_0=\eta(\xi_0)>0$,

c) $\lim\limits_{\xi\to\xi_1}\eta(\xi)=+\infty$.

It should be noted that studying the solution of equation (11) is equivalent to studying the solution of equation (1), that is, each of them satisfies the following inequalities on the interva $[\eta_0,+\infty)$:

$$y(\eta)>0,\quad Ly\neq 0$$

The solution $y(\eta)$ of equation (11) tends to a finite limit y_0 as $\eta\to+\infty$. Moreover, the following hold:

$$\lim_{\eta\to+\infty}\frac{e^{-\eta}}{a-e^{-\eta}}=0,\ \lim_{\eta\to+\infty}y'=0.$$

For the previously introduced functions $b_i(\eta), i=0,1$, we now compute the following limits as $\eta\to+\infty$:

$$b_0 = \lim_{\eta\to+\infty} b_0(\eta) = -m,$$

$$b_1 = \lim_{\eta\to+\infty} b_1(\eta) = \frac{\gamma_1^{1-p}}{C^{\sigma+p-2}p},$$

$$y(\eta) = y_0 + o(1),$$

$$\lim_{\eta\to+\infty} Ly = \lim_{\eta\to+\infty} y^{\sigma}\left|\frac{dy}{d\eta} - my\right|^{p-2}\left(\frac{dy}{d\eta} - my\right) = -m^{p-1}y_0{}^{\sigma+p-1} + o(1)$$

.

According to equation (11), the derivative of Ly has a limit as $\eta \to +\infty$, and this limit is equal to zero. Therefore, the following condition is necessary:

$$\lim_{\eta\to+\infty}\left[b_0(\eta)Ly + b_1(\eta)\left(\frac{dy}{d\eta} - my\right)\right] = 0$$

The self-similar solution of equation (5) has the following asymptotic behavior near the point $x \to \int_0^{\tau(t)} \rho(s)ds + \left(a^{\frac{p-1}{p}}\left(\tau(t)\right)^{\frac{1}{p}}\right)$ (where $\tau(t)$-is the function defined above):

$$u_A = y_0\left[\frac{1}{n\cdot m}\left(\frac{1}{p}\right)^{\frac{1}{p-1}}\right]^m\left(a - \left[\frac{x - \int_0^{\tau(t)}\rho(s)ds}{\left[\int\frac{1}{\rho(t)}dt\right]^{\frac{1}{p}}}\right]^n\right)_+^m\left(1 + o(1)\right) \quad (12)$$

Here, $0 < y_0 < +\infty$, and y_0 is a root of a certain nonlinear algebraic equation:

$$m^{p-1}y_0{}^{\sigma+p-2} - \frac{n^{1-p}}{C^{\sigma+p-2}\cdot p} = 0 \quad (13)$$

If the initial condition (2) is given by $u(0,x) = E_0\delta(x)$, where $\delta(x)$ is the Dirac delta function, $E_0 = \int_R \left(a - \xi^n\right)_+^m d\xi$.

$$F(n,m) = \int_R \left(a - \xi^n\right)_+^m d\xi \quad (14)$$

According to the function condition $(d)_+ = \max(0,d)$, equation (14) can be rewritten as follows:

$$F(n,m) = \int_{-a^{\frac{1}{n}}}^{a^{\frac{1}{n}}} \left(a - \xi^n\right)_+^m d\xi \quad (15)$$

Let n be an even number. Then we continue equation (15) as follows:

$$F(n,m)=2\cdot\int_0^{a^{\frac{1}{n}}}\left(a-\xi^n\right)_+^m d\xi=\left|\xi^n=z\right|=\frac{2}{n}\cdot\int_0^{1/a^n}(a-z)^m z^{\frac{1}{n}-1}dz=\left|u=\frac{z}{a}\right|=$$

$$=\frac{2}{n}a^{m+\frac{1}{n}}\int_0^{1/a^{n+1}}(1-u)^m u^{\frac{1}{n}-1}du=\frac{2}{n}a^{m+\frac{1}{n}}\mathrm{B}\left(\frac{1}{n},m+1\right)=\frac{2}{n}a^{m+\frac{1}{n}}\frac{\Gamma\left(\frac{1}{n}\right)\Gamma(m+1)}{\Gamma\left(m+1+\frac{1}{n}\right)}$$

(16)

According to the initial condition and equation (16), the following equality holds:

$$E_0=\frac{2}{n}a^{m+\frac{1}{n}}\frac{\Gamma\left(\frac{1}{n}\right)\Gamma(m+1)}{\Gamma\left(m+1+\frac{1}{n}\right)}.$$

From this, we obtain the following expression for a:

$$a(E_0)=\left(\frac{nE_0}{2}\frac{\Gamma\left(m+1+\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n}\right)\Gamma(m+1)}\right)^{\frac{1}{m+\frac{1}{n}}}.$$

6. Numerical Analysis

We construct a finite difference scheme for problem (1)–(2). In the domain Ω, with step size h, we define a uniform grid ω_h:

$$\omega_h=\left\{x_i=ih,\ h=\frac{b}{n},\ i=0,1,\ldots,n,\ hn=b\right\}$$

and for time, we introduce a uniform time grid with step size τ, forming the mesh ω_τ,

$$\omega_\tau=\left\{t_j=j\tau,\ \tau=\frac{T}{m},\ j=0,1,\ldots,m,\ \tau m=T\right\}.$$

Using the balance method, we replace problem (1)–(2) with a finite difference scheme and obtain a finite difference problem with an error of order $O\left(h^2+\tau\right)$.

By applying the balance method, we construct an implicit finite difference scheme for problem (1)–(2), resulting in a finite difference problem that includes an approximation error:

$$\begin{cases} \rho(t_j)\dfrac{y_i^{j+1}-y_i^j}{\tau}=\dfrac{1}{h^2}a_{i+1}(y^{j+1})\left|\dfrac{y_{i+1}^{j+1}-y_i^{j+1}}{h}\right|^{p-2}(y_{i+1}^{j+1}-y_i^{j+1})- \\ -\dfrac{1}{h^2}a_i(y^j)\left|\dfrac{y_i^{j+1}-y_{i-1}^{j+1}}{h}\right|^{p-2}(y_i^{j+1}-y_{i-1}^{j+1})-\rho(t_j)\dfrac{y_{i+1}^{j+1}-y_{i-1}^{j+1}}{2h} \\ i=1,2.....,n-1;\ j=0,1,.....,m-1, \\ y_i^0=u_0(x_i),\ i=0,1.....,n; \\ y_0^j=u_A(x,t_j)\big|_{x=a},\ j=1,2.....,m; \\ y_n^j=u_A(x,t_j)\big|_{x=b},\ j=1,2.....,m; \end{cases} \tag{17}$$

Here, to compute $a(y)$, one of the following formulas is used:

a) $a_i(y)=K\left(\dfrac{y_{i-1}^{j+1}+y_i^{j+1}}{2}\right)$,

b) $a_i(y)=\dfrac{K\left(y_{i-1}^{j+1}\right)+K\left(y_i^{j+1}\right)}{2}$.

The system of algebraic equations (17) is nonlinear with respect to y^{j+1}. o solve the nonlinear system, we use the iteration method and obtain the following:

$$\begin{aligned} \rho(t_j)\frac{y_i^{s+1,j+1}-y_i^{s,j}}{\tau}&=\frac{1}{h^2}a_{i+1}\left(y^{s,j+1}\right)\left|\frac{y_{i+1}^{s,j+1}-y_i^{s,j+1}}{h}\right|^{p-2}\left(y_{i+1}^{s+1,j+1}-y_i^{s+1,j+1}\right)- \\ -\frac{1}{h^2}a_i\left(y^{s,j}\right)&\left|\frac{y_i^{s,j+1}-y_{i-1}^{s,j+1}}{h}\right|^{p-2}\left(y_i^{s+1,j+1}-y_{i-1}^{s+1,j+1}\right)-\rho(t_j)\frac{y_{i+1}^{s+1,j+1}-y_{i-1}^{s+1,j+1}}{2h} \end{aligned} \tag{18}$$

where $s=0,1,2,....$

The finite difference scheme (18) is linear with respect to y_i^{s+1}. The value of y, specifically y_i^{s+1}, is initialized using the value from the previous time step as the initial iteration: $y^{0,j+1}=y^j$. In the iterative scheme, a given accuracy criterion is set,

and the iteration process continues until the following condition is satisfied:

$$\max_{0\le i\le n}\left|{}^{s+1}y_i-{}^{s}y_i\right|<\varepsilon .$$

In equation (18), we introduce the following notations:

$${}^{s}A_i=\frac{\tau}{h^2}a_i\left({}^{s}y_i^{j}\right)^{\sigma}\cdot\left|\frac{{}^{s}y_i^{j+1}-{}^{s}u_{i-1}^{j+1}}{h}\right|^{p-2}+\frac{\rho\left(t_j\right)}{2h}$$

$$,\ {}^{s}B_i=\frac{\tau}{h^2}a_{i+1}\left({}^{s}y_{i+1}^{j}\right)\cdot\left|\frac{{}^{s}y_{i+1}^{j+1}-{}^{s}y_i^{j+1}}{h}\right|^{p-2}-\frac{\rho(t_j)}{2h}$$

,

$${}^{s}C_i={}^{s}A_i+{}^{s}B_i+\rho\left(t_j\right),\ {}^{s}F_i={}^{s}y_i^{j},\ \mathrm{s}=0,1,2\ldots,$$

$$\mathrm{i}=1,2,\ldots,\mathrm{n}.$$

Let us agree on introducing the following notations:

$$y^{j}=y,\ y^{j+1}=\bar{y}.$$

The finite difference equation can be written in the following form:

$${}^{s}A_i\,{}^{s+1}\bar{y}_{i-1}-{}^{s}C_i\,{}^{s+1}\bar{y}_i+{}^{s}B_i\,{}^{s+1}\bar{y}_{i+1}=-{}^{s}F_i,$$

$$i=1,2,\ldots,n-1. \qquad (19)$$

To numerically solve the system of equations (19), the shooting method is used [1,4–7]. From (17) and the boundary conditions, we obtain the following:

$$\bar{y}_0=\phi_1\left(t_{j+1}\right),\ \bar{y}_n=\phi_2\left(t_{j+1}\right).$$

$$\bar{y}_i=\alpha_i\left(\beta_i+\bar{y}_{i+1}\right),$$

(20)

Here, α_i and β_i are unknown coefficients, and they are computed as follows [4–7]:

$$\begin{cases}\alpha_i=\dfrac{A_i}{C_i-\alpha_{i-1}B_i}\\ \beta_i=\dfrac{B_i\alpha_{i-1}\beta_{i-1}+F_i}{A_i}\end{cases},\ i=\overline{2,n}.$$

(21)

The quantities α_1 and β_1 are determined from equations (10) and (9) when $\mathrm{i}=1$:

$$\bar{y}_1=\alpha_1\left(\beta_1+y_2\right),$$

$$\bar{y}_1=\frac{A_1y_2+B_1\bar{y}_0+F_1}{C_1}.$$

From this,

$$\begin{cases}\alpha_1=\dfrac{A_1}{C_1}\\ \beta_1=\dfrac{B_1\varphi_1+F_1}{A_1}\end{cases}$$

(22)

Thus, for $i=\overline{1,n}$ the coefficients α_i and β_i are determined.

The solution of system (17) is obtained using the recurrence formula (20). The values of $\bar{y}_n$ are derived from the boundary condition of equation (17):

$$\begin{cases} \overline{y}_n = \phi_2 \\ \overline{y}_{n-1} = \alpha_{n-1}\left(\beta_{n-1} + y_n\right) \\ \vdots \\ \overline{y}_i = \alpha_i\left(\beta_i + \overline{y}_{i+1}\right) \\ \vdots \end{cases} \tag{23}$$

To apply the shooting method, it is sufficient to require that the coefficients of system (19) satisfy the following conditions:

$$A_i \neq 0, \quad B_i \neq 0, \quad |C_i| \geq |A_i| + |B_i|, \; i = \overline{1,n}. \tag{24}$$

For the iterative scheme (18), the shooting method provides a stable and unique solution. By using the Python programming language, we obtain numerical results and visualize the process based on these results.

We illustrate the graphs of problem (1)–(2) for different values of the parameters.

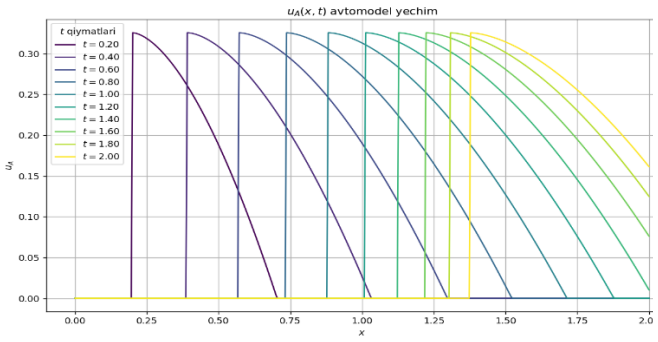


Figure .1 One-dimensional case (

$p = 2.7,\ \sigma = 1.5,\ \alpha = 1.1,\ a = 1.0,\ T_0 = 1.2$**)**

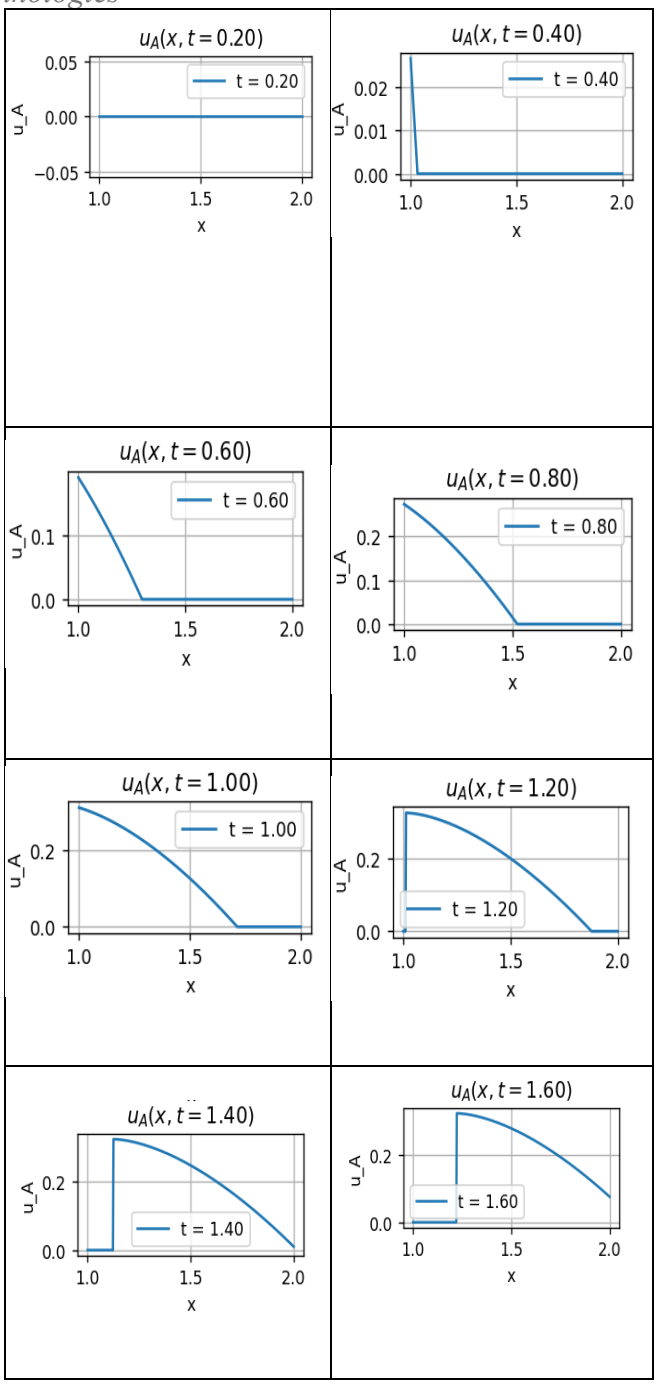


Figure 2. Graphical representation of the solution (

$p = 2.5,\ \sigma = 1.2,\ \alpha = 1.2,\ a = 1.0,\ T_0 = 1.0$**)**

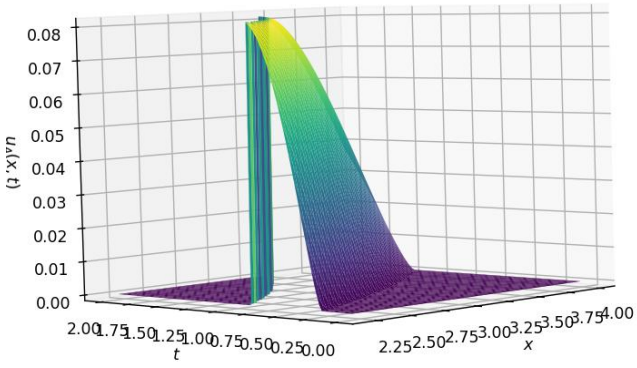


Figure 3. Two-dimensional case (

$p = 2.5,\ \sigma = 0.5,\ \alpha = 1.0,\ a = 1.0,\ T_0 = 1.6$**)**

Conclusion

In this article, we studied the Cauchy problem for a parabolic-type equation with time-dependent density, which models nonlinear processes such as diffusion, filtration, and heat propagation in an inhomogeneous medium. The global solution of the problem was obtained, and estimates for it were provided. Furthermore, through self-similar analysis, the asymptotic behavior of compactly supported solutions was established.

It is known that solving problems of this type often presents many challenges. One of the most critical issues is how to choose an appropriate initial approximation when solving such problems numerically. If the self-similar solution we derived is used as an initial approximation, it ensures faster convergence of the iterative process. The obtained numerical results, reflected in the graphs, illustrate the nonlinear heat propagation process in a medium with time-dependent exponentially varying density.

References

[1] А.А. Самарский Теория разностных схем. Издательство «НАУКА» Москва, 1977 г., 656 стр.

[2] Aripov M., Sadullaeva Sh.A. Computer modeling of nonlinear diffusion processes. Tashkent Universit. 2020. p 672

[3] Самарский А.А., Галактионов В.А., Курдюмов С.П., Михайлов А.П. Режимы с обострением в задачах для квазилинейных параболических уравнений. – М.: Наука, 1987, 480с.

[4] Самарский А.А., Гулин А.В.Численные методы. —М.: Наука.1989.432 с.

[5] Самарский А.А., Николаев Е.С. Методы решения сеточных уравнений. —М.: Наука.1978.592 с.

[6] Исроилов М. Ҳисоблаш методлари.2-қисм.Т.: Iqtisod-Moliya 2008.,322 b.

[7] Basti B., Benhamidouche N. Global existence and blow-up of generalized self-similar solutions to nonlinear degenerate diffusion equation not in divergence form. Appl. Math. E-Notes, 2020.20, 367-387.

[8] Aripov M., Matyakubov A.S., Khasanov, J. O., Bobokandov M. M. Mathematical modeling of double nonlinear problem of reaction diffusion in not divergent form with a source and variable density. In Journal of Physics: Conference Series, 2021, December. (Vol. 2131, No. 3, p. 032043).

[9] Begulov U.U., Khaydarov A.T., Salimov J.I. Mathematical modeling of the double nonlinear exsponential inhomogeneous density heat dissipation process. ACTA NUUz. Exact sciences 2024 2/1. 34-42 p.

[10] Begulov U.U., Salimov J.I. Chiziqsiz reaksiya-diffuziyali epidemiya masalasini matematik modellashtirish. Educational Research in Universal Science, 5/2023. 511-516 p.

[11] Begulov U.U., Khaydarov A.T. Eksponensial o'zgaruvchan zichlikli chiziqsiz muhitda issiqlik tarqalish jarayonini matematik modellashtirish. O'zMU xabarlari, Toshkent — 2023. 246-253 p.

[12] Begulov U.U., Khaydarov A.T., M.I. Davronova The Cauchy problem for a double nonlinear parabolic equation with variable density and nonlinear source. "QarDU Xabarlari", 2024 3(2). 95-102 p.

[13] Aripov M.M., Sadullaeva Sh.A. To Properties of Solutions to Reaction-diffusion Equation with Double Nonlinearity with Distributed Parameters. Journal of Siberian Federal University. Mathematics & Physics.2013; 6(2):157–167.

[14] Galaktionov V.A., King J.R. On the behaviour of blow-up interfaces for an inhomogeneous filtration equation. Journal of Applied Mathematics. 1996;57(1):53–77.

[15] Martynenko A.V., Tedeev A.F., Shramenko V.N. Cauchy problem for a degenerate parabolic equation with inhomogeneous density and source in class slowly tending to zero initial functions. Izvestiya. Rossiiskoi Akademii Nauk. Seriya Matematicheskaya.2012;76(3):139–156.

[16] Martynenko A.V., Tedeev A.F. Cauchy Problem for a quasilinear parabolic equation with source and inhomogeneous density. Computational Mathematics and Mathematical Physics. 2007;47(2):238-248.